Int. J. Heat Mass Transfer. Vol. 18, pp. 495–502. Pergamon Press 1975. Printed in Great Britain

ON A DIRECT VARIATIONAL METHOD FOR NONLINEAR HEAT TRANSFER

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(*Received* 2 *April* 1974)

Abstract—In this paper it is shown, that the partial nonlinear differential equation can be reduced to the variational problem. By means of the variational technique based on the Kantorovich method, a nonlinear boundary value problem can be reduced to the set of the ordinary differential equations. The accuracy of the method is estimated by comparing the solutions of problems solved using the variational method and the other method. In this paper the method for the construction of the trial functions is also presented. Three examples are included to illustrate the above method.

NOMENCLATURE

- T, temperature;
- c, specific heat;
- λ, conductivity of solid;
- W, heat sources;
- q, heat-transfer rate per unit area;
- α, coefficient of heat transfer;
- τ, time;
- F, surface of the considered body.

1. INTRODUCTION

AT THE present time the majority of the heat-transfer problems are to be solved by use of variational calculus. The main reason for this is that the variational approach considerably reduces the number of calculations without decreasing the accuracy of the results. The existing variational solutions, involving the unsteady heat flow with the heat sources, have been obtained using simplified mathematical models. It has been done by the assumption that the thermal properties are considered to be constants. This simplified problem is shown in [2–5].

A similar assumption may be found in [8]. The heat-transfer problems, with the thermal properties being temperature dependent can be found in recent literature, for instance, in the papers given by Biot [11], Djukic and Vojanovic [9].

Biot has expressed the heat flow problem in a quasi-variational form using the concept of thermal potential, dissipation function and generalized thermal forces. The characteristic of Biot analysis is that, the variational integral does not exist, thus, this principle is the Galerkin concept rather than the variational one.

Djukic and Vojanovic have expressed the problem as a variational principle of Hamiltonian type. It is worth noticing that the above considerations are not correct, because they do not preserve all the rules of variational calculus. Introducing the parameter λ, which tends to zero after the process of variation is completed, causes the variational integral not to correspond to the initial differential equation. Also the initial condition is not pointed out in the paper [9]. Due to that, the proposed method may be treated as of Biot's type.

The formulation described in the present paper satisfies the conditions ncessary for the existence of an extremum of variational integral. The approximate solution is found by use of Kantorovich method.

This method is based upon the assumption that the parameters A_i of the function $T_n(\xi, \zeta, \eta, \tau)$, being the approximate solution of the form

$$\sum_{0}^{n} A_i \varphi_i(\xi, \eta, \zeta),$$

are unknown functions of the time τ at the first stage of the consideration and have to be determined at the next stage from the relevant set of the ordinary differential equations. Such a solution is, in general, much more accurate than that obtained by the classical Ritz method with the same set of trial functions and the same number of parameters A_i.

Improved accuracy is achieved owing to the fact that the class of the functions used for the solution is of much more general form, than the class of the function in the case of the "pure" Ritz method.

The application of the more general mathematical model will enable us to extend the variational method for the case of a number of engineering problems concerning nuclear reactors, chemical apparatus etc.

2. BASIC EQUATIONS AND THE VARIATIONAL PRINCIPLE

2.1. *The first form of variational integral*

We shall be concerned with a solid body with heat sources, the intensity of which is $W(T)$ and is a known function of the temperature. The coefficient of heat conduction is assumed to be a function of the temperature.

The similar assumption involves the specific heat $C(T)$. It will also be assumed that the total surface F of the body is a sum of the component surfaces and the different boundary conditions are prescribed on different component surfaces.

The temperature field in the region considered is determined by the following differential equation of parabolic type:

$$\operatorname{div}[\lambda(T)\operatorname{grad}T]+W[T]=C(T)\cdot\frac{\partial T}{\partial\tau} \qquad (1)$$

the boundary conditions on the component surfaces are:

$$[T-T_k]_{F_1}=0, \quad \tau>0 \qquad (2)$$

$$\left[\lambda(T)\cdot\frac{\partial T}{\partial n}+q(T)\right]_{F_2}=0, \quad \tau>0 \qquad (3)$$

$$\left[\lambda(T)\cdot\frac{\partial T}{\partial n}+\alpha(T^m-T_a^m)\right]_{F_3}=0, \quad \tau>0 \qquad (4)$$

and the initial condition:

$$T=T_p(\xi,\eta,\zeta), \quad \tau=0. \qquad (5)$$

and it thus completes the proof.

It can easily be shown that this integral has a minimum, because the Legendre condition

$$\frac{\partial}{\partial T_\xi}\left(\frac{\partial L}{\partial T_\xi}\right)=\lambda^2>0, \quad \frac{\partial}{\partial T_\eta}\left(\frac{\partial L}{\partial T_\eta}\right)=\lambda^2>0,$$

$$\frac{\partial}{\partial T_\zeta}\left(\frac{\partial L}{\partial T_\zeta}\right)=\lambda^2>0,$$

is satisfied, where

$$L\equiv\frac{\lambda^2}{2}(\operatorname{grad}T)^2-\int_{T_0}^{T}W(\vartheta)\cdot\lambda(\vartheta)\cdot d\vartheta+\frac{\partial T}{\partial\tau}\int_{T_0}^{T}C(\vartheta)\cdot\lambda(\vartheta)\cdot d\vartheta$$

and

$$T_\xi=\frac{\partial T}{\partial\xi}, \quad T_\eta=\frac{\partial T}{\partial\eta}, \quad T_\zeta=\frac{\partial T}{\partial\zeta}.$$

It will be shown that the equation (1) together with the boundary conditions (2–4) and initial condition (5) can be reduced to the variational problem

$$Y\langle T\rangle\equiv\int_0^t\left\{\int_V\left[\frac{\lambda^2(T)}{2}\cdot(\operatorname{grad}T)^2-\int_{T_0}^{T}W(\vartheta)\cdot\lambda(\vartheta)\cdot d\vartheta+F(x,y,z,t)\int_{T_0}^{T}C(\vartheta)\cdot\lambda(\vartheta)\cdot d\vartheta\right]dV\right\}d\tau$$
$$+\int_0^t\left\{\int_{F_2}\left[\int_{T_0}^{T}q(\vartheta)\cdot\lambda(\vartheta)\cdot d\vartheta\right]dF\right\}d\tau+\alpha\int_0^t\left\{\int_{F_3}\left[\int_{T_0}^{T}(\vartheta^m-T_a^m)\lambda(\vartheta)\cdot d\vartheta\right]dF\right\}d\tau+Z\Big|_{\tau=0}=\text{minimum} \qquad (6)$$

where $\dfrac{\partial T}{\partial\tau}=F(x,y,z,t)$ T_0 – reference temperature

and

$$Z\Big|_{\tau=0}=\frac{1}{2}\int_V(T-T_p)^2\,dV$$

verify the validity of this relation, let us write the first variation of the expression (6), that is

$$\delta Y\langle T\rangle\equiv\int_0^t\left\{\int_V\left[(\operatorname{grad}T)^2\cdot\lambda(T)\cdot\omega(T)\cdot\delta T+\lambda^2(T)\cdot\operatorname{grad}T\cdot\operatorname{grad}\delta T-W(T)\cdot\lambda(T)\cdot\delta T\right.\right.$$
$$\left.\left.+\frac{\partial T}{\partial\tau}\cdot C(T)\cdot\lambda(T)\cdot\delta T\right]dV\right\}d\tau+\int_0^t\left\{\int_{F_2}\left[q(T)\cdot\lambda(T)\cdot\delta T\right]dF\right\}d\tau+\int_0^t\left\{\int_V\alpha\left[(T^m-T_a^m)\lambda(T)\cdot\delta T\right]dF\right\}d\tau$$
$$+\int_V\left[(T-T_p)\delta T\right]_{\tau=0}dV=0 \qquad (7)$$

where

$$W(T)=\frac{d\lambda}{dT}.$$

Then applying of the Gauss–Ostrogradsky relation we find:

$$\delta Y\langle T\rangle\equiv\int_0^t\left\{\int_V\left[\frac{\partial T}{\partial\tau}\cdot C(T)-\operatorname{div}(\lambda(T)\cdot\operatorname{grad}T)-W(T)\right]\lambda(T)\cdot\delta T\cdot dV\right\}d\tau$$
$$+\int_0^t\left\{\int_{F_2}\left[\lambda(T)\cdot\mathbf{n}\cdot\operatorname{grad}T+q(T)\right]\lambda(T)\cdot\delta T\cdot dF\right\}d\tau+\int_0^t\left\{\int_{F_3}\left[\lambda(T)\cdot\mathbf{n}\cdot\operatorname{grad}T+\alpha(T^m-T_a^m)\right]\lambda(T)\cdot\delta T\cdot dF\right\}d\tau$$
$$+\int_V\left[(T-T_p)\delta T\right]\Big|_{\tau=0}dV=0 \qquad (8)$$

It is seen directly that the first integrand of equation (8) is obtained from the differential equation (1) and the remaining integrands from the boundary conditions (3–4), and from initial condition (5).

In order to increase the accuracy of the computation, the trial functions should be so selected that the boundary conditions are satisfied. It enables to reduce the relevant surface integrals. In the considered case we restrict the variation in such a way that $\delta T/F_1 = 0$. When the thermal properties are of the form:

$$\begin{aligned} \lambda &= \lambda_1(\xi,\eta,\zeta)\cdot\lambda_2(T) \\ C &= C_1(\xi,\eta,\zeta)\cdot C_2(T) \\ W &= W_1(\xi,\eta,\zeta)\cdot W_2(T) \\ q &= q_1(\xi,\eta,\zeta)\cdot q_2(T) \end{aligned} \tag{9}$$

the similar equation, as equation (8), may be derived from the variational principle.

It can shown, that in the case of nonlinear partial differential equation both formulation—the Galerkin and variational one—reduces the problem to the system of ordinary differential equation. The systems have different form and are depending upon the method to be used. It should be noted that in the linear case the Galerkin method will yield the same results as variational method.

2.2. *The second form of variational integral*

In many engineering problems the variational integral related to the differential equation (1) and boundary condition (2–5) may be expressed in another form. This is of particular use in the case of the Neumann problem. In the subsequent considerations, we shall assume that the differential equation (1) has the boundary condition expressed by equation (3) only.

Introducing a new function expressed by relation:

$$\Theta = \frac{1}{\lambda_0}\int_0^{\vartheta} \lambda(t)\cdot \mathrm{d}t \tag{10}$$

where

$$\vartheta = T - T_0$$

we obtain

$$\Theta = H(\vartheta). \tag{11}$$

It is assumed that there exists the inverse function

$$\vartheta = F(\Theta). \tag{12}$$

Substituting it into (1), (3), (5), we obtain

$$\lambda_0 \operatorname{div}(\operatorname{grad}\Theta) + W[F(\Theta)] = C[F(\Theta)]\cdot\frac{\partial[F(\Theta)]}{\partial\tau} \tag{13}$$

$$\lambda_0\frac{\partial\Theta}{\partial n} + q[F(\Theta)]\bigg|_F = 0,\quad \tau > 0 \tag{14}$$

$$F(\Theta) = F_p(\Theta),\quad \tau = 0. \tag{15}$$

In this case the variational integral may be written in the form:

$$Y\langle\Theta\rangle \equiv \int_0^t\left\{\int_V\left[\frac{\lambda_0}{2}(\operatorname{grad}\Theta)^2 - \int_0^{\Theta} W[F(\omega)]\,\mathrm{d}\omega + G(x,y,x,t)\int_0^{\Theta} C[F(\omega)]\cdot\mathrm{d}\omega\right]\mathrm{d}V\right\}\mathrm{d}\tau + \int_0^t\left\{\int_F\left[q[F(\omega)]\,\mathrm{d}\omega\right]\mathrm{d}F\right\}\mathrm{d}\tau + \frac{1}{2}\int_V [F(\Theta)-F_p]^2_{t=0}\cdot\mathrm{d}V = \text{minimum} \tag{16}$$

where

$$\frac{\partial F}{\partial\tau} = G(x,y,z,t)$$

The first variation δY of the relation (16) is given by

$$\delta Y\langle\Theta\rangle \equiv \int_0^t\left\{\int_V\left[\lambda_0 \operatorname{grad}\Theta\cdot\operatorname{grad}\delta\Theta - W[F(\Theta)]\cdot\delta\Theta + \frac{\partial F(\Theta)}{\partial\tau}\cdot C[F(\Theta)]\cdot\delta\Theta\right]\mathrm{d}V\right\}\mathrm{d}\tau + \int_0^t\left\{\int_F q[F(\Theta)]\cdot\delta\Theta\cdot\mathrm{d}F\right\}\mathrm{d}\tau + \int_V\left\{[F(\Theta)-F_p]\,\delta\Theta\right\}\bigg|_{t=0}\cdot\mathrm{d}V = 0. \tag{17}$$

Then, applying the Gauss-Ostrogradsky relation, we have

$$\delta Y\langle\Theta\rangle \equiv \int_0^t\left\{\int_V\left[\frac{\partial F(\Theta)}{\partial\tau}\cdot C[F(\Theta)] - \lambda_0\operatorname{div}(\operatorname{grad}\Theta) - W[F(\Theta)]\right]\delta\Theta\cdot\mathrm{d}V\right\}\mathrm{d}\tau + \int_0^t\left\{\int_F\left[\lambda_0\cdot\mathbf{n}\cdot\operatorname{grad}\Theta + q[F(\Theta)]\right]\delta\Theta\cdot\mathrm{d}F\right\}\mathrm{d}\tau + \int_V\left\{[F(\Theta)-F_p]\,\delta\Theta\right\}\bigg|_{t=0}\mathrm{d}V = 0. \tag{18}$$

It can be easily shown that the functional (16) satisfies the Euler and Legendre conditions for a minimum.

In technical problems the function $\lambda(\vartheta)$ may be written in the following form

$$\lambda(\vartheta) = \lambda_0 - \lambda_1 \vartheta. \tag{19}$$

Using relation (10), we find

$$\Theta = \frac{1}{\lambda_0}\int_0^{\vartheta}(\lambda_0 - \lambda_1 \omega)\,\mathrm{d}\omega = \vartheta - \frac{\lambda_1}{2\lambda_0}\vartheta^2 \tag{20}$$

in this case the inverse function is of the form

$$\vartheta = \frac{\lambda_0}{\lambda_1}\left(1 - \sqrt{\left[1 - \frac{2\lambda_1 \Theta}{\lambda_0}\right]}\right). \tag{21}$$

3. COMPUTATION FLOW. PARTIAL SOLUTIONS

In the subsequent consideration, we shall describe the computation procedure by the Kantorovich method.

In this method it is assumed that the approximate solution is a linear combination of sequence of functions:

$$T_n = \sum_0^n A_i(\tau)\cdot\varphi_i(\xi,\eta,\zeta) \quad [i = 0, 1, \ldots, n]. \tag{22}$$

The function $\varphi_i(\xi,\eta,\zeta)$ is assumed as a known, where A_i are undetermined functions of a single variable τ.

The selection of the trial functions φ_i is, of course, the crucial point in any procedure of this type. It is up to the ingenuity of the analyst to construct the trial function so that a maximum of information can be extracted with a minimum of computation.

In order to overcome this difficulty we shall follow the method suggested in [8].

The basic idea consists in dividing the initial problem described by equation (1) into N sub-problems. The sub-problems are expressed by the set of the partial differential equations of the elliptical type.

$$\operatorname{div}(\operatorname{grad} T) = \sum_{m,n,l=0}^{N} A_{m,n,l}\cdot\xi^m\cdot\eta^n\cdot\zeta^l \quad [m, n, l = 1, 2, \ldots, N]. \tag{23}$$

where the right hand sides of equation (23) express the power series of the location variables. The power series may contain undetermined coefficients $A_{m,n,l}$ to be obtained later from the relevant set of differential equations.

To obtain partial solutions, we proceed as follows: the first partial solution we get solving the equation (23) with the first term of power series, that is $m = n = l = 0$, and with the nonhomogeneous boundary conditions. The next partial solution can be obtained using the second term of power series and taking the homogeneous boundary condition. In this fashion it is possible to obtain as many partial solutions as is desired. The substitution of this partial solution into equation (8) and the subsequent integration will yield a set of first-order ordinary differential equations. The system of ordinary differential equations have certain initial values, which corresponds to initial condition expressed by equation (5).

4. ESTIMATION OF THE INITIAL VALUES

In the subsequent consideration we shall describe the computation procedure to find the initial values for a set of ordinary differential equations.

In our case the method for obtaining the initial values consists in minimizing the functional:

$$\int_V [T_p - T_n]^2\cdot\mathrm{d}V = \text{minimum}. \tag{24}$$

The right minimum is zero and yields the solution. When the preceding technique is set up for a nonlinear problem, we get a set of nonlinear algebraic equations for obtaining the initial values.

The true roots are selected by means of the principle of minimum. Then if several approximate solutions are available, the corresponding values for minimum may be used as a basis for comparison: if the value of minimum is lower then the approximation is better.

5. NUMERICAL EXAMPLE

As a numerical example illustrating the computation procedure, let us consider the problem of unsteady heat conduction in an infinite plate of thickness δ, the initial temperature being zero. Because of the bounding planes, $\xi = 0$ is insulated and the temperature of the other $\xi = 1$ is constant in time $\tau > 0$ and equal to 1.

This problem will be solved in approximate manner in several variants, and the results will be compared. In the case of the approximate solution procedure or, more precisely, in the case of the estimation of the trial function, we can apply the procedure described in the previous section.

We shall assume that the coefficient of temperature conduction is dependent upon the temperature according to the relation

$$\kappa = \kappa_0(1 - \kappa_1\vartheta) \tag{25}$$

where $\kappa_0, \kappa_1 = \text{const.}$

The heat equation, valid in the region considered, has the form

$$\frac{\partial}{\partial\xi}\left[(1-\kappa_1\vartheta)\frac{\partial\vartheta}{\partial\xi}\right] = \frac{\partial\vartheta}{\partial F_0} \tag{26}$$

the boundary conditions

$$\lambda\cdot\left.\frac{\partial\vartheta}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\vartheta\right|_{\xi=1} = 1 \tag{27}$$

and the initial condition

$$\vartheta_p(\xi,0) = 0, \quad 0 \leqslant \xi \leqslant 1, \quad F_0 = 0. \tag{28}$$

5.1. *Galerkin method*

In the Galerkin method, it is assumed that the approximate solution has the form

$$\vartheta_n = 1 + A_0(\xi^2-1) + A_1(\xi^3-1). \tag{29}$$

The approximate solution, being composed of two terms, is obtained using the procedure described in Section 3 (*vide* also [8]).

Making use of the Galerkin method, we obtain the relation

$$\int_0^1 \left\{ \frac{\partial \vartheta_n}{\partial F_0} - \frac{\partial}{\partial \xi}\left[(1-\kappa_1\vartheta_n)\frac{\partial \vartheta_n}{\partial \xi} \right] \right\} \delta\vartheta_i \cdot \mathrm{d}\xi = 0 \qquad [i = 0, 1] \tag{30}$$

where

$$\delta\vartheta_0 = \frac{\partial \vartheta_n}{\partial A_0}\delta A_0 = (\xi^2-1)\,\delta A_0$$

$$\delta\vartheta_1 = \frac{\partial \vartheta_n}{\partial A_1}\delta A_1 = (\xi^3-1)\,\delta A_1 .$$

The substitution of the approximate solution into equation (30) and the subsequent integration will yield a set of the first order ordinary differential equations:

$$\begin{aligned} &1.\ -\tfrac{4}{3}(1-\kappa_1)A_0 - \tfrac{3}{2}(1-\kappa_1)A_1 - \tfrac{8}{15}\kappa_1 A_0^2 - \tfrac{7}{6}\kappa A_0 A_1 \\ &\qquad - \tfrac{9}{14}\kappa A_1^2 - \tfrac{8}{15}\dot{A}_0 - \tfrac{7}{12}\dot{A}_1 = 0, \\ &2.\ -\tfrac{3}{2}(1-\kappa_1)A_0 - \tfrac{9}{5}(1-\kappa_1)A_1 - \tfrac{\kappa_1}{2}A_0^2 - \tfrac{81}{70}\kappa_1 A_0 A_1 \\ &\qquad - \tfrac{27}{40}\kappa_1 A_1^2 - \tfrac{7}{12}\dot{A}_0 - \tfrac{9}{14}\dot{A}_1 = 0. \end{aligned} \tag{31}$$

We can find the initial values for a set of ordinary differential equations minimizing the functional:

$$\int_0^1 [\vartheta_p - \vartheta_n^{(0)}]^2 \cdot \mathrm{d}\xi = \text{minimum} \tag{32}$$

where

$$\vartheta_n(0) = 1 + A_0(0)(\xi^2-1) + A_1(0)(\xi^3-1).$$

Hence

$$A_0(0) = -3{\cdot}4608; \qquad A_1(0) = 4{\cdot}30719. \tag{33}$$

The numerical results are represented in Table 1 and Fig. 1.

Table 1

κ	F_0	ξ										
		0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0	0·01	0·0940	0·0733	0·0244	−0·0330	−0·0787	−0·0931	−0·0564	0·0515	0·2503	0·5599	1·0000
0	0·02	0·0530	0·0397	0·0102	−0·0207	−0·0377	−0·0256	0·0306	0·1462	0·3362	0·6157	1·0000
0	0·05	0·0019	0·0026	0·0087	0·0265	0·0622	0·1219	0·2121	0·3386	0·5081	0·7264	1·0000
0	0·10	0·0413	0·0504	0·0777	0·1237	0·1887	0·2732	0·3775	0·5020	0·6470	0·8129	1·0000
0	0·20	0·2220	0·2326	0·2621	0·3092	0·3723	0·4499	0·5405	0·6424	0·7540	0·8737	1·0000
0	0·40	0·5232	0·5296	0·5483	0·5779	0·6175	0·6658	0·7218	0·7844	0·8524	0·9246	1·0000
0	1·00	0·8915	0·8930	0·8972	0·9040	0·9130	0·9240	0·9367	0·9510	0·9664	0·9829	1·0000
0·10	0·01	0·0970	0·0758	0·0258	−0·0335	−0·0812	−0·0974	−0·0619	0·0454	0·2447	0·5562	1·0000
0·10	0·02	0·0580	0·0439	0·0121	−0·0218	−0·0420	−0·0330	0·0210	0·1356	0·3265	0·6094	1·0000
0·10	0·05	0·0082	0·0076	0·0102	0·0234	0·0540	0·1092	0·1959	0·3213	0·4924	0·7163	1·0000
0·10	0·10	0·0404	0·0482	0·0725	0·1144	0·1751	0·2557	0·3576	0·4817	0·6294	0·8018	1·0000
0·10	0·20	0·2075	0·2167	0·2459	0·2909	0·3526	0·4295	0·5202	0·6238	0·7391	0·8649	1·0000
0·10	0·40	0·4941	0·5007	0·5197	0·5503	0·5913	0·6418	0·7006	0·7669	0·8396	0·9176	1·0000
0·10	1·00	0·8683	0·8701	0·8752	0·8833	0·8942	0·9075	0·9229	0·9401	0·9590	0·9790	1·0000
0·20	0·01	0·1000	0·0783	0·1051	−0·0340	−0·0837	−0·1016	−0·0675	0·0393	0·2392	0·5526	1·0000
	0·02	0·0630	0·0481	0·0140	−0·0228	−0·0465	−0·0405	0·0112	0·1250	0·3168	0·6032	1·0000
	0·05	0·0138	0·0117	0·0110	0·0193	0·0447	0·0951	0·1785	0·3027	0·4758	0·7055	1·0000
	0·10	0·0405	0·0470	0·0679	0·1053	0·1611	0·2375	0·3365	0·4603	0·6107	0·7899	1·0000
	0·20	0·1934	0·2024	0·2288	0·2722	0·3320	0·4075	0·4981	0·6032	0·7224	0·8548	1·0000
	0·40	0·4650	0·4716	0·4908	0·5219	0·5839	0·6160	0·6775	0·7474	0·8251	0·9095	1·0000
	1·00	0·8412	0·8432	0·8493	0·8588	0·8720	0·8879	0·9064	0·9273	0·9500	0·9744	1·0000
0·30	0·01	0·1040	0·0818	0·0289	−0·0336	−0·1188	−0·1051	−0·0724	0·0337	0·2339	0·5491	1·0000
	0·02	0·0680	0·0522	0·0159	−0·0241	−0·0510	−0·0481	0·0013	0·1141	0·3070	0·5967	1·0000
	0·05	0·0206	0·0172	0·0127	0·0160	0·0360	0·0815	0·1613	0·2841	0·4590	0·6947	1·0000
	0·10	0·0417	0·0468	0·0640	0·0963	0·1468	0·2185	0·3144	0·4375	0·5908	0·7773	1·0000
	0·20	0·1807	0·1888	0·2131	0·2537	0·3107	0·3841	0·4740	0·5805	0·7036	0·8434	1·0000
	0·40	0·4360	0·4424	0·4615	0·4925	0·5348	0·5882	0·6519	0·7256	0·8085	0·9001	1·0000
	1·00	0·8098	0·8123	0·8194	0·8306	0·8460	0·8648	0·8868	0·9117	0·9391	0·9687	1·0000

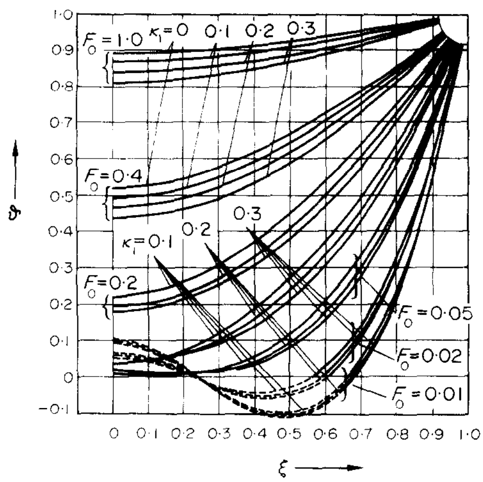

FIG. 1. The temperature distribution in the flat plate (Galerkin method).

5.2. *Variational method. The first form of variational integral*

In this case, we obtain the relations following the existence of an extremum of the variational integral considered

$$\delta Y \equiv \int_0^1 \left\{\frac{\partial \vartheta_n}{\partial F_0} - \frac{\partial}{\partial \xi}\left[(1-\kappa_1 \vartheta_n)\frac{\partial \vartheta_n}{\partial \xi}\right]\right\}(1-\kappa_1\vartheta_n) \times \delta\vartheta_i \cdot \mathrm{d}\xi = 0 \quad [i = 0, 1] \quad (34)$$

where

$$\delta\vartheta_0 = \frac{\partial \vartheta_n}{\partial A_0}\delta A_0 = (\xi^2 - 1)\delta A_0$$

$$\delta\vartheta_1 = \frac{\partial \vartheta_n}{\partial A_1}\delta A_1 = (\xi^3 - 1)\delta A_1 .$$

The approximate solution has the same form as in equation (29).

Making use of equation (34), a similar computation is carried out as in Section (5.1), yielding the set of differential equations

$$\begin{aligned} 1.\ & \tfrac{8}{15}\dot{A}_0(1-\kappa_1) + \tfrac{7}{12}(1-\kappa_1)\dot{A}_1 + \kappa_1 \\ & \times\left[\tfrac{16}{35}\cdot \dot{A}_0 A_0 + \tfrac{59}{120}A_0\dot{A}_1 + \tfrac{59}{120}\dot{A}_0 A_1 + \tfrac{67}{126}\dot{A}_1 A_1\right] \\ & + \tfrac{4}{3}A_0(1-\kappa_1)^2 + \tfrac{3}{2}A_1(1-\kappa_1)^2 + \kappa_1(1-\kappa_1) \qquad (35) \\ & \times\left[\tfrac{24}{15}A_0^2 + \tfrac{10}{3}A_0A_1 + \tfrac{9}{5}A_1^2\right] \\ & + \kappa_1^2\left[\tfrac{111}{60}\cdot A_0^2A_1 + \tfrac{602}{315}A_0A_1^2 + \tfrac{64}{105}\cdot A_0^3 + \tfrac{189}{280}\cdot A_1^3\right] = 0; \end{aligned}$$

$$\begin{aligned} 2.\ & \tfrac{7}{12}\dot{A}_0(1-\kappa_1) + \tfrac{9}{14}\cdot\dot{A}_1(1-\kappa_1) + \kappa_1 \\ & \times\left[\tfrac{59}{120}\dot{A}_0 A_0 + \tfrac{67}{126}A_0\dot{A}_1 + \tfrac{67}{126}\cdot\dot{A}_0 A_1 + \tfrac{81}{140}\dot{A}_1 A_1\right] \\ & + \tfrac{3}{2}A_0(1-\kappa_1)^2 + \tfrac{9}{5}A_1(1-\kappa_1)^2 + \kappa_1(1-\kappa_1) \\ & \times\left[\tfrac{5}{3}A_0^2 + \tfrac{126}{35}A_0A_1 + \tfrac{81}{40}A_1^2\right] \\ & + \kappa_1^2\left[\tfrac{602}{315}A_0^2A_1 + \tfrac{567}{280}\cdot A_1^2A_0 + \tfrac{37}{60}A_0^3 + \tfrac{81}{110}A_1^3\right] = 0. \end{aligned} \quad (36)$$

The initial values are obtained in a similar way as in Section 5.1, that is

$$\int_0^1 [\vartheta_p - \vartheta_n]^2\,\mathrm{d}\xi = \text{minimum} \qquad \begin{aligned} A_0(0) &= -3{\cdot}4608; \\ A_1(0) &= 4{\cdot}30719. \end{aligned}$$

The numerical results are represented in Table 2.

5.3. *Variational method. The second form of variational integral*

In the third variant of our problem, the relations derived in Section 2.2 will be used.

Substituting in (26) the relation (21), we obtain the heat equation in the region considered in the form

$$\frac{\partial^2\Theta}{\partial\xi^2} - \frac{1}{\sqrt{(1-2\kappa_1\Theta)}}\cdot\frac{\partial\Theta}{\partial F_0} = 0 \qquad (37)$$

the boundary conditions

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\Theta\right|_{\xi=1} = 1 - \frac{\kappa_1}{2} \qquad (38)$$

and initial condition

$$\Theta_p(\xi, 0) = 0, \ 0 \leqslant \xi \leqslant 1, F_0 = 0 . \qquad (39)$$

The approximate solution, being obtained using the procedure described in Section 3, has the form

$$\Theta_n = \left(1 - \frac{\kappa_1}{2}\right) + A_0(\xi^2-1) + A_1(\xi^3-1). \qquad (40)$$

Making use of the relation (18), we obtain

$$\int_0^1 \left[\frac{\partial^2\Theta_n}{\partial\xi^2} - \frac{1}{\sqrt{(1-2\kappa_1\Theta_n)}}\cdot\frac{\partial\Theta_n}{\partial F_0}\right]\cdot\varphi_i\cdot\mathrm{d}\xi = 0 \qquad [i = 0{\cdot}1] \quad (41)$$

where

$$\varphi_i = \frac{\partial\Theta_n}{\partial A_i}.$$

The initial values for a set of ordinary differential equations can be obtained as follows:

$$\int_0^1 \left[\left(1 - \left.\sqrt{(1-2\kappa_1\Theta_n)}\right|_{F_0=0}\right)\cdot \left.\frac{(\xi^2-1)}{\sqrt{(1-2\kappa_1\Theta_n)}}\right|_{F_0=0}\right]\cdot\mathrm{d}\xi = 0 \quad (42)$$

$$\int_0^1 \left[\left(1 - \left.\sqrt{(1-2\kappa_1\Theta_n)}\right|_{F_0=0}\right)\cdot \left.\frac{(\xi^3-1)}{\sqrt{(1-2\kappa_1\Theta_n)}}\right|_{F_0=0}\right]\cdot\mathrm{d}\xi = 0. \quad (43)$$

The numerical results are represented in Table 3 and Fig. 2.

Table 2

κ_1	F_0	ξ										
		0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·00
$\kappa_1 = 0·1$	0·02	0·0580	0·0438	0·0117	−0·0224	−0·0430	−0·0343	0·0195	0·1341	0·3253	0·6086	1·00
	0·05	0·0085	0·0077	0·0101	0·0228	0·0530	0·1077	0·1942	0·3196	0·4909	0·7153	1·00
	0·10	0·0406	0·0484	0·0724	0·1140	0·1744	0·2548	0·3564	0·4805	0·6283	0·8011	1·00
	0·20	0·2072	0·2169	0·2450	0·2904	0·3521	0·4289	0·5197	0·6233	0·7386	0·8640	1·00
	0·40	0·4939	0·5005	0·5195	0·5501	0·5910	0·6415	0·7004	0·7667	0·8394	0·9175	1·00
	1·00	0·8683	0·8700	0·8751	0·8833	0·8941	0·9074	0·9229	0·9401	0·9590	0·9790	1·00
$\kappa_1 = 0·2$	0·02	0·0633	0·0482	0·0138	−0·0235	−0·0475	−0·0420	0·0095	0·1231	0·3152	0·6021	1·00
	0·05	0·0148	0·0126	0·0113	0·0190	0·0438	0·0936	0·1765	0·3006	0·4738	0·7043	1·00
	0·10	0·0409	0·0472	0·0678	0·1047	0·1601	0·2361	0·3348	0·4584	0·6091	0·7889	1·00
	0·20	0·1931	0·2020	0·2284	0·2716	0·3312	0·4066	0·4971	0·6023	0·7215	0·8543	1·00
	0·40	0·4645	0·4711	0·4903	0·5213	0·5633	0·6155	0·6770	0·7470	0·8247	0·9093	1·00
	1·00	0·8411	0·8431	0·8492	0·8589	0·8718	0·8878	0·9063	0·9272	0·9499	0·9743	1·00
$\kappa_1 = 0·3$	0·02	0·0683	0·0524	0·0159	−0·0244	−0·0517	−0·0491	0·0001	0·1128	0·3058	0·5959	1·00
	0·05	0·0211	0·0175	0·0126	0·0155	0·0350	0·0800	0·1594	0·2822	0·4573	0·6936	1·00
	0·10	0·0420	0·0469	0·0639	0·0959	0·1460	0·2174	0·3130	0·4360	0·5894	0·7764	1·00
	0·20	0·1805	0·1886	0·2128	0·2532	0·3100	0·3833	0·4732	0·5796	0·7029	0·8430	1·00
	0·40	0·4357	0·4421	0·4611	0·4920	0·5344	0·5877	0·6515	0·7251	0·8081	0·8999	1·00
	1·00	0·8097	0·8122	0·8192	0·8306	0·8458	0·8646	0·8867	0·9116	0·9390	0·9688	1·00

Table 3

κ_1	F_0	ξ										
		0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·00
$\kappa_1 = 0·1$	0·02	0·0595	0·0456	0·0139	−0·0201	−0·0414	−0·0347	0·0150	0·1240	0·3098	0·5931	1·00
	0·05	0·0067	0·0064	0·0100	0·0238	0·0545	0·1087	0·1933	0·3158	0·4845	0·7088	1·00
	0·10	0·0384	0·0466	0·0715	0·1140	0·1751	0·2556	0·3569	0·4802	0·6272	0·7998	1·00
	0·20	0·2063	0·2161	0·2445	0·2903	0·3523	0·4292	0·5200	0·6236	0·7389	0·8647	1·00
	0·40	0·4943	0·5009	0·5201	0·5507	0·5917	0·6421	0·7009	0·7671	0·8397	0·9177	1·00
	1·00	0·8700	0·8717	0·8767	0·8848	0·8955	0·9086	0·9239	0·9409	0·9595	0·9793	1·00
$\kappa_1 = 0·2$	0·02	0·0642	0·0498	0·0168	−0·0195	−0·0438	−0·0416	0·0024	0·1047	0·2849	0·5697	1·00
	0·05	0·0097	0·0087	0·0103	0·0208	0·0472	0·0963	0·1755	0·2935	0·4605	0·6897	1·00
	0·10	0·0354	0·0428	0·0655	0·1048	0·1620	0·2385	0·3364	0·4580	0·6064	0·7853	1·00
	0·20	0·1908	0·2002	0·2275	0·2719	0·3323	0·4081	0·4987	0·6035	0·7222	0·8545	1·00
	0·40	0·4665	0·4732	0·4926	0·5238	0·5659	0·6180	0·6793	0·7489	0·8261	0·9101	1·00
	1·00	0·8480	0·8500	0·8558	0·8651	0·8775	0·8928	0·9105	0·9304	0·9522	0·9755	1·00
$\kappa_1 = 0·3$	0·02	0·0689	0·0540	0·0197	−0·0186	−0·0460	−0·0479	−0·0097	0·0856	0·2592	0·5437	1·00
	0·05	0·0127	0·0111	0·0107	0·0182	0·0404	0·0845	0·1582	0·2710	0·4350	0·6680	1·00
	0·10	0·0328	0·0394	0·0601	0·0962	0·1494	0·2218	0·3158	0·4349	0·5837	0·7687	1·00
	0·20	0·1763	0·1852	0·2114	0·2540	0·3125	0·3867	0·4764	0·5819	0·7037	0·8425	1·00
	0·40	0·4394	0·4461	0·4657	0·4972	0·5399	0·5932	0·6565	0·7292	0·8110	0·9014	1·00
	1·00	0·8251	0·8273	0·8339	0·8444	0·8586	0·8760	0·8963	0·9192	0·9443	0·9714	1·00

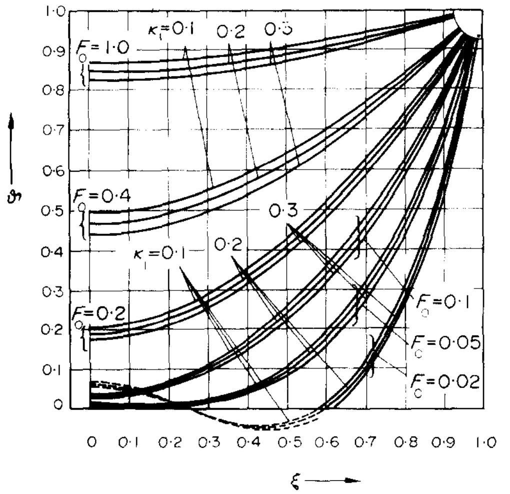

FIG. 2. The temperature distribution in the flat plate (the second form of variational integral).

6. FINAL REMARKS

Usually the unsteady heat flow problems are considered by the assumption that the thermal properties are constant. This assumption has been omitted in the present paper.

It has been shown that this type of problem can be treated as a variational problem. Two different forms of the variational integral have been constructed. The choice of a suitable form is dependent upon the boundary conditions. The calculation flow has been illustrated on the basis of three numerical examples.

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